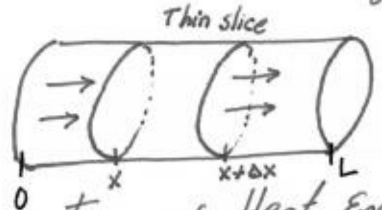
CHAPTER I

Derivation of the Heat Equation

Unsteady heat conduction in a rod laterally perfectly insulated



Conservation Law: Conservation of Heat Energy.

$$\boxed{\text{Rate of change of heat energy inside slice per unit of time.}} = \boxed{\text{Net flow of heat energy across boundaries per unit of time}} + \boxed{\text{Heat energy generated or taken away inside slice per unit of time}} \quad (1)$$

$e(x,t)$: Thermal energy density, $[e] = \frac{E}{L^3}$

$\phi(x,t)$: Heat Flux, $[\phi] = \frac{E}{L^2 \times t}$ $\phi > 0 \Rightarrow$ Heat energy flowing right; $\phi < 0 \Rightarrow$ Heat energy flowing left.

$Q(x,t)$: Heat Sources or Sinks $[Q] = \frac{E}{L^3 \times t}$

Conservation Law (1) is equivalent to (assuming thermal energy varies only from one cross-section to another).

$$\boxed{\frac{\partial}{\partial t}\left(e(\bar{x},t)\right) A\,\Delta x = \phi(x,t)A - \phi(x+\Delta x,t)A + Q(\bar{x},t)\,A\,\Delta x} \quad (1.2)$$

Dividing by $A\Delta x$ and letting $\Delta x \to 0$ ($x < \bar{x} < x+\Delta x$)

$$\boxed{\frac{\partial e}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t)} \quad (1.3)$$

Equation (1.2) is obtained from

$$\frac{d}{dt}\left[\int_x^{x+\Delta x} e(x,t)A\,dx\right] = \phi(x,t)A - \phi(x+\Delta x,t)A + \int_x^{x+\Delta x} Q(x,t)A\,dx \qquad (2.1)$$

Recall <u>mean value theorem for integrals</u>

f conts. on $[a,b]$, then there is c in (a,b)
Such that
$$\int_a^b f(x)dx = f(c)\,(b-a).$$

then (2.1) transforms into

$$\frac{d}{dt}\left[e(\bar{x},t)A\,\Delta x\right] = \phi(x,t)A - \phi(x+\Delta x,t)A + Q(\bar{\bar{x}},t)A\,\Delta x$$

where $\bar{x}, \bar{\bar{x}}$ are in $(x, x+\Delta x)$

Then dividing by $A\,\Delta x$

$$\frac{\partial e}{\partial t}(\bar{x},t) = \frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x} + Q(\bar{\bar{x}},t)$$

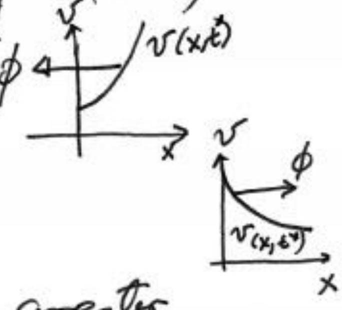
Taking limit $\Delta x \to 0$

$$\boxed{\frac{\partial e}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t)} \qquad (2.2)$$

Now, we introduce

I) Fourier's law of Heat conduction:

$$\boxed{\phi(x,t) = -K_0(x)\frac{\partial v}{\partial x}(x,t).} \qquad (2.3)$$



Explain properties:

1) Flux From Hotter to Colder $\longrightarrow$

2) Flow is greater when $\frac{\partial v}{\partial x}$ is greater (temperature Variations)

3) No flow is $\frac{\partial v}{\partial x} = 0$.

4) Flow depends on material.

II) Relationship between heat Energy density and temperature.

$$\boxed{e(x,t) = c(x)\,\rho(x)\,v(x,t)} \qquad (2.4)$$

So $[e] = \frac{E}{L^3}$, $[c] = \frac{E}{M * T}$, $[v] = T$, $[\rho] = \frac{M}{L^3}$

So the physical units match.

Substituting (2.3) and (2.4) into (2.2)

$$C(x)\,\rho(x)\,\frac{\partial v}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial v}{\partial x}\right) + Q(x,t)$$

Dividing by $C(x)\rho(x)$

$$\boxed{\frac{\partial v}{\partial t}(x,t) = \frac{1}{C(x)\rho(x)}\,\frac{\partial}{\partial x}\left(K_0(x)\frac{\partial v}{\partial x}(x,t)\right) + \frac{Q(x,t)}{C(x)\rho(x)}} \quad (2.5)$$

If C, ρ and K_0 are constants

then

$$\frac{\partial v}{\partial t}(x,t) = \frac{K_0}{C\rho}\,\frac{\partial^2 v}{\partial x^2}(x,t) + \frac{1}{C\rho}Q(x,t)$$

or

$$\boxed{\frac{\partial v}{\partial t} = \kappa\frac{\partial^2 v}{\partial x^2} + \tilde{Q} \qquad 0 < x < L,\ t > 0} \quad (2.6)$$

$$\kappa \equiv \frac{\overset{\text{Thermal conductivity}}{K_0}}{C\rho} : \text{Thermal diffusivity}$$

Therefore, a good law (or relationship) is given by

$$\boxed{\phi(x,t) = -K_0(x)\,\frac{\partial u}{\partial x}(x,t)} \qquad (*)$$

Explain why the previous (4) Conditions are Satisfied.

<u>Exercise</u>: (add to the homework in section 1.2)

Find the physical units of $K_0(x)$. in terms of Energy (E), Length (L), Temperature (T), time (t).

<u>Diffusion of Chemical Pollutant:</u>

$u(x,t)$: density of chemical $[u] = \dfrac{\text{Amount of chemical}}{L^3}$

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x}$$

$$\left\{\begin{array}{l} \phi(x,t) = -k\,\dfrac{\partial u}{\partial x} \\ \text{Fick's Law of diffusion} \end{array}\right.$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}}$$

Same as Heat conduction.

k → chemical diffusivity.

Chemicals spread out from regions of high concentration to regions of low concentration.

1.3 Boundary Conditions.

A) Prescribed Temperature:

$$V(a,t) = A(t)., \qquad \underline{a = 0 \text{ or } L}$$

Prescribed flux or
B) Insulated boundary:

$$\phi(a,t) = -K_0(a)\frac{\partial V}{\partial x}(a,t) = f(t).$$

$$\text{or simply } \boxed{\frac{\partial V}{\partial x}(a,t) = g(t)}$$

if $g(t) \equiv 0$ insulated

C) Newton's Law of Cooling:

$$\phi(0,t) = -K_0(0)\,\frac{\partial V}{\partial x}(0,t) = -H\left[V(0,t) - \underbrace{V_B(t)}_{\text{Surrounding rod temperature}}\right].$$

(I) If rod temperature > Surrounding at $x = 0$

$$\Rightarrow \quad V(0,t) - V_B(t) > 0 \Rightarrow \phi(0,t) < 0$$

heat flows out of rod

(II) If rod temp < Surrounding. at $x = 0$ $\Rightarrow V(0,t) - V_B(t) < 0 \Rightarrow \phi(0,t) > 0$

Heat flows from the surrounding to the rod.

Plus an initial Condition:

$$v(x,0) = f(x). , \quad 0 < x < L.$$

The set of equations:

$$\begin{cases} \dfrac{\partial v}{\partial t} = k \dfrac{\partial^2 v}{\partial x^2} + \dfrac{Q}{c\rho} , & 0<x<L \\ & t>0 \\ v(0,t) = A, & t>0 \\ \dfrac{\partial v}{\partial x}(L,t) = 0, & t>0 \\ v(x,0) = f(x), & 0<x<L \end{cases}$$

form an initial boundary Value problem (IBVP).

there are several analytic techniques to obtain its solution